1 Sets Versus Classes

Before we begin working with spaces, let's take just a minute to discuss some of the ways that set theory interacts with category theory (and infinity-category theory), specifically in terms of "small" and "large" categories.

To better understand what we mean by small and large categories, we must first understand the distinction between a class and a set. Intuitively, we know a set is just a (potentially infinite) collection of objects, such as $\{a, b, c\}, \{\}$, and $\mathbb{R}$. Sets are also special in that we require them to obey the axioms of ZFC; for us to call a collection a "set", it must abide by certain rules. Doing so allows us to work with sets in a coherent and structured way, and use them as a formal basis for mathematics.

For example, one such rule is that if A and B are sets, their union $A \cup B$ must also be a set. Another rule that will be crucial to the discussion below is that if A is a set, then the powerset $\mathscr{P}(A)$ is also a set, and is strictly larger than A; that is, there is no injection from $\mathscr{P}(A)$ to A. We often write this fact as $|A| < |\mathscr{P}(A)|$.

However, there are some collections that we say are "too large" to be sets.

❏ **Example:** Suppose for sake of contradiction there was a *set of all sets*, denoted V. Then as a consequence of the axioms of ZFC, $\mathscr{P}(V)$ would also be a set, and $|V| < |\mathscr{P}(V)|$. However, since V contains every set, any set in $\mathscr{P}(V)$ must also be in V, so $|\mathscr{P}(V)| \leq |V|$. This is a contradiction!

❏ **Example:** We have a similar issue if we define G to be the set of all groups. Since any set can be considered a group by giving it the free group structure, the existance of G would imply the existance of V. Therefore, such a set G cannot exist.

So what went wrong? One perspective on the issue presented in [2] is as a conflict between *extensibility* and *universality*. Based on the axioms of ZFC, one could say that sets are *extensible*; given any set A, we can make a new set that is strictly larger than A (such as by taking the powerset of A). Because sets are *extensible*, they cannot be *universal* (there cannot be a set that contains everything).[1]

However, it is possible and often advantageous to refer to these "universal" collections (like when we want to talk about the categories **Set** and **Grp**). To navigate this issue, we carefully refer to these universal collections as *classes*. In doing so, we acknowledge that they are not sets, and therefore do not obey the axioms of ZFC (in particular, they are not extensible). Since we make no "rules" that these collections must follow, we can refer to them without inviting paradox.

Let's tie this back to category theory! Recall that we say a category C is small if $\mathrm{ob}(C)$ is the same size as some set. If $\mathrm{ob}(C)$ is so universal that no such set can exist in ZFC, we say that

[1] It is also noteworthy that if V were to be a set, then $V \in V$. While this fact violates the axioms of ZFC, it is not inherently paradoxical for a set to contain itself. Also, there do exist conceptions of set theory which accommodate a "set of all sets" by restricting extensibility. ZFC is not one such set theory.

ob(C) is class-sized. If we have a category C so that ob(C) is class-sized but for any two objects X, Y the collection $\mathrm{Hom}_C(X, Y)$ is set-sized, we say C is locally small. If ob(C) is class-sized and $\mathrm{Hom}_C(X, Y)$ is class-sized for some $X, Y \in \mathrm{ob}(C)$, we say C is large.[2]

The following definition will be useful in constructing some examples.

Definition: Let C be a category. We let Free(C) denote the free category (sometimes called the path category) generated on the directed graph underlying C. The objects of Free(C) are the same as the objects of C. The morphisms of Free(C) are all finite paths along morphisms in C.

Note that unique paths are unique morphisms in Free(C), even when their composition in C yields the same arrow. For example, if C is the category consisting only of a single object and its identity morphism, then Free(C) is the one-object category with a countably infinite number of distinct morphisms.

❑ **Exercise:** Recall that **Set** is a locally small category. Show that Free(**Set**) is large.

2 Issues of Size in Higher Topos Theory

In Higher Topos Theory Section 1.2.15 [1], Lurie puts forward three possible ways we can approach the issue of universal collections in regular and higher category theory:

1. Make a distinction between "small" and "large" categories using a set-theoretic device such as Grothendieck Universes or a formal definition for "class".
2. Work exclusively with small categories, and "mirror the distinction between large and small by keeping careful track of relative sizes."
3. Decide to ignore these set-theoretic technicalities. ☺

He opts for the first approach, specifically by utilizing Grothendieck Universes. Grothendieck Universes are intimately tied to inaccessible cardinals, which we'll discuss below.

Here's a quick and informal definition of cardinal for those who may be unfamiliar: we often talk about the *cardinality* of a set, which really just means its size. A *cardinal* is a special set that is the "representative" of its size. For any possible size a set can be, there's exactly one cardinal κ that has the same size, and "represents" things of that size; the same way the number 4 "represents" the size of all sets with 4 elements. For example, the cardinal representing size 2 is $\{\{\varnothing\}, \varnothing\}$. The cardinal representing countably infinite size is $\mathbb{N}$, which set theorists often write as $\aleph_0$ to emphasize we're thinking about $\mathbb{N}$ as a cardinal.

Definition: An inaccessible cardinal is a cardinal κ satisfying the following conditions:

1. $\kappa > \aleph_0$ (κ is uncountable)
2. κ cannot be expressed as the union of fewer than κ-many sets, each with cardinality less than κ (κ is *regular*).
3. If $|A| < \kappa$, then $|\mathscr{P}(A)| < \kappa$ (κ is a *strong limit cardinal*).

[2] When we call class-sized collections "too large to be sets", it could seem to imply that there exists a set S for which sets smaller than S are sets, but anything larger than S is a class. No such set exists.

Such a cardinal κ is "inaccessible" in the sense that it cannot be "accessed" by taking unions and powersets, which are the two tools at our disposal for making bigger sets in ZFC. Indeed, we could think of κ as a limit on our "extensibility", creating a ceiling through which our set operations cannot breach.

To gain some intuition on regular and strong limit cardinals (conditions 2 and 3 above), one can think of the natural numbers $\aleph_0$. Because the union of finitely-many finite sets its itself finite, $\aleph_0$ is regular. Additionally, since the powerset of any finite set is still a finite set, $\aleph_0$ is also a strong limit cardinal. Thus $\aleph_0$ would be an inaccessible cardinal were it not for condition 1. [3]

We'll follow Lurie's lead and take the following proposition as an axiom.

✧**Axiom:** For any cardinal λ, there exists a strongly inaccessible cardinal $\kappa > \lambda$. This implies the existence of an infinite chain of inaccessible cardinals $\kappa_0 < \kappa_1 < \kappa_2 < \ldots$.

It's worth noting that the existence of strongly inaccessible cardinals cannot be proven by ZFC, and cannot be proven by ZFC to be consistent with ZFC. Lurie however emphasizes that assuming their existence here is simply a convenient means to dealing with size issues in category theory, and that none of the results in Higher Topos Theory "depend on this assumption in an essential way."

❀**Definition:** For an inaccessible cardinal κ, the Grothendieck Universe $\mathscr{U}(\kappa)$ is the collection of all sets with cardinality less than κ.

So why bother doing this? A Grothendieck Universe $\mathscr{U}(\kappa_i)$ gives us a sufficiently large universe to do ZFC set theory comfortably in, without fear of building sets that "extend outside" the universe. However, we might still use the components of that universe to define a category too *large* to fit within our universe. No matter! In many cases, we can simply jump then to a larger universe, namely $\mathscr{U}(\kappa_{i+1})$. In this new, expanded universe, our category is now *small* by comparison, and so we can continue on our merry way. Because we have an inexhasutable list of inaccessible cardinals, we can repeat this trick as many times as we like.

Like Lurie, Markus Land also uses this Grothendieck Universe approach for dealing with issues of size in his textbook *Introduction to Infinity Categories.* For the purposes of that text, he cites the first three Grothendieck Universes $\mathscr{U}(\kappa_0)$, $\mathscr{U}(\kappa_1)$, and $\mathscr{U}(\kappa_2)$ as being sufficient.

❏ **Exercise:** Let C be a category that is small in $\mathscr{U}(\kappa_i)$ (its object set and all its Hom sets have cardinality less than κ_i). Determine whether Free(C) is small in $\mathscr{U}(\kappa_{i+1})$.

[3] Some texts do forgo condition 1 and consider $\aleph_0$ to be the first inacessible cardinal. From this perspective, ZFC's Infinity Axiom is a "large cardinal" axiom!

References

[1] Jacob Lurie. *Higher Topos Theory (AM-170)*. Princeton University Press, 2009.

[2] Adam Rieger. Paradox, zf, and the axiom of foundation. In David DeVidi, Michael Hallett, and Peter Clark, editors, *Logic, Mathematics, Philosophy, Vintage Enthusiasms: Essays in Honour of John L. Bell*, pages 171–187. Springer, 2011.